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Industrial Process Control
MDP 454

If you have a smart project, you can say "I'm an engineer"

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Lecture 5

Staff boarder

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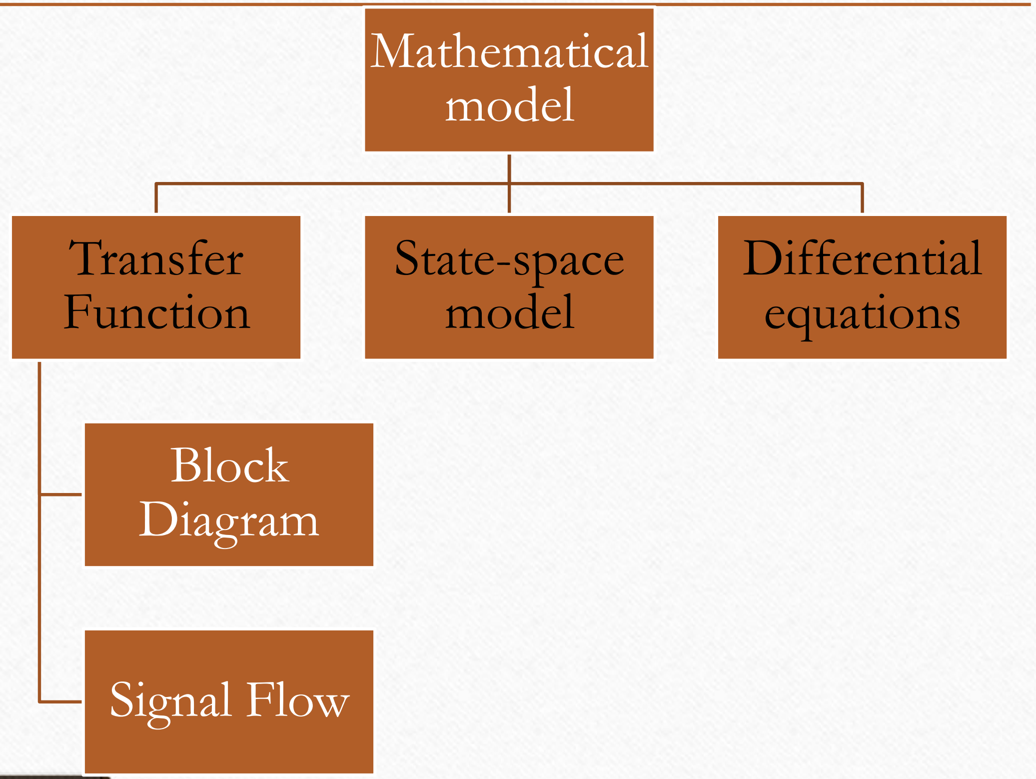
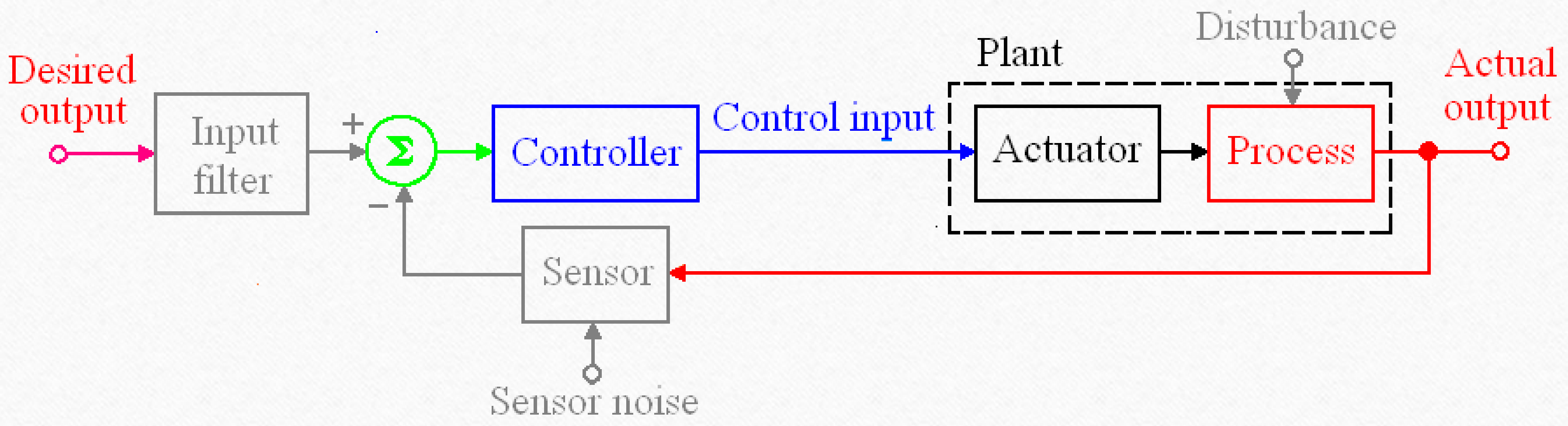
- Lecture aims:
  - Understand the Block reduction techniques
  - Identify the transfer function
  - Be aware by modeling multiple technique

Mathematical Modeling

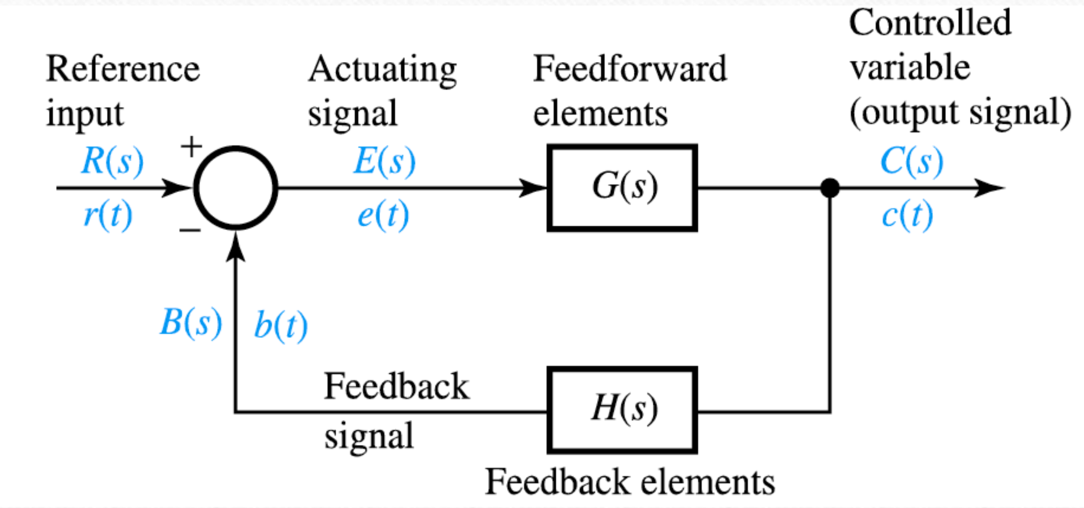
- Transfer Function



Component Block Diagram



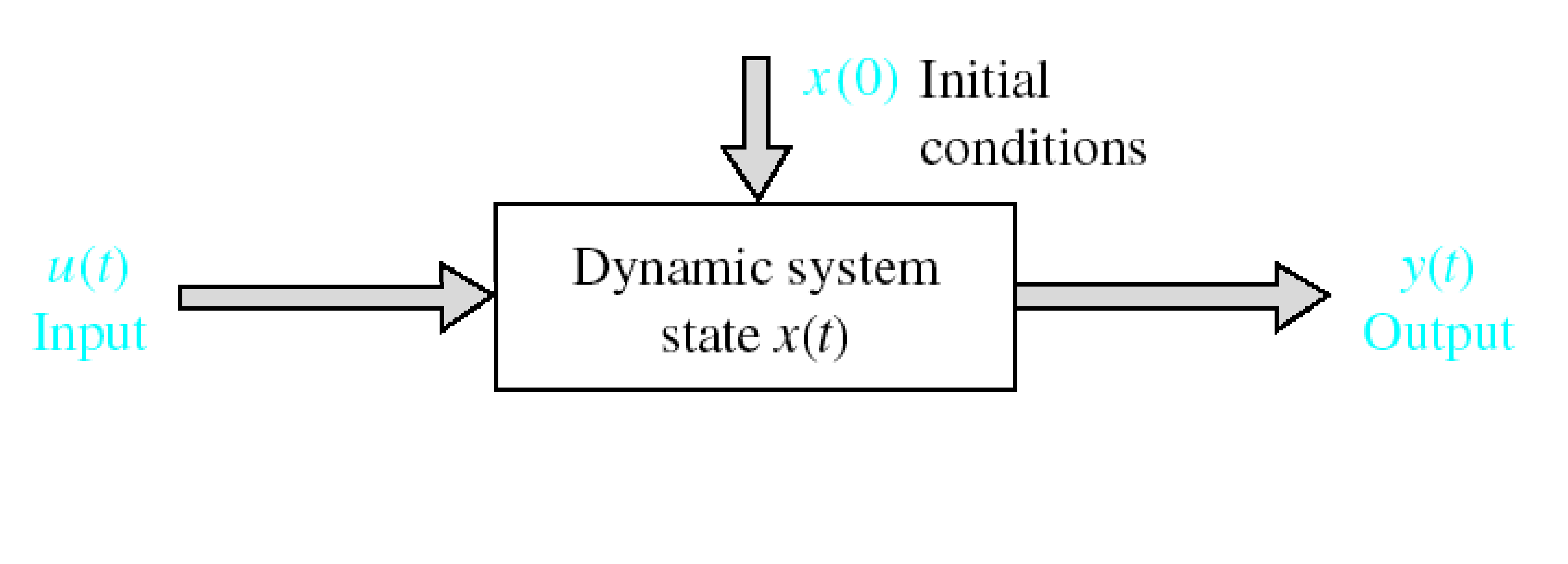
Component Block Diagram



$R(s)$ Reference input
$C(s)$ Output signal (controlled variable)
$B(s)$ Feedback signal $= H(s)C(s)$
$E(s)$ Actuating signal (error) $= [R(s) - B(s)]$
$G(s)$ Forward path transfer function or open-loop transfer function $= C(s)/E(s)$
$M(s)$ Closed-loop transfer function $= C(s)/R(s) = G(s)/[1 + G(s)H(s)]$
$H(s)$ Feedback path transfer function
$G(s)H(s)$ Loop gain

$$\frac{E(s)}{R(s)} = \text{Error-response transfer function} \quad \frac{1}{1 + G(s)H(s)}$$

The general form of a dynamic system



The general form of a dynamic system

State Space Equations

- **State equations** is a description which relates the following four elements: input, system, state variables, and output

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

Matrix A has dimensions nxn and it is called the **system** matrix, having the general form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Matrix B has dimensions nxm and it is called the **input** matrix, having the general form

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

Matrix C has dimensions pxn and it is called the **output** matrix, having the general form

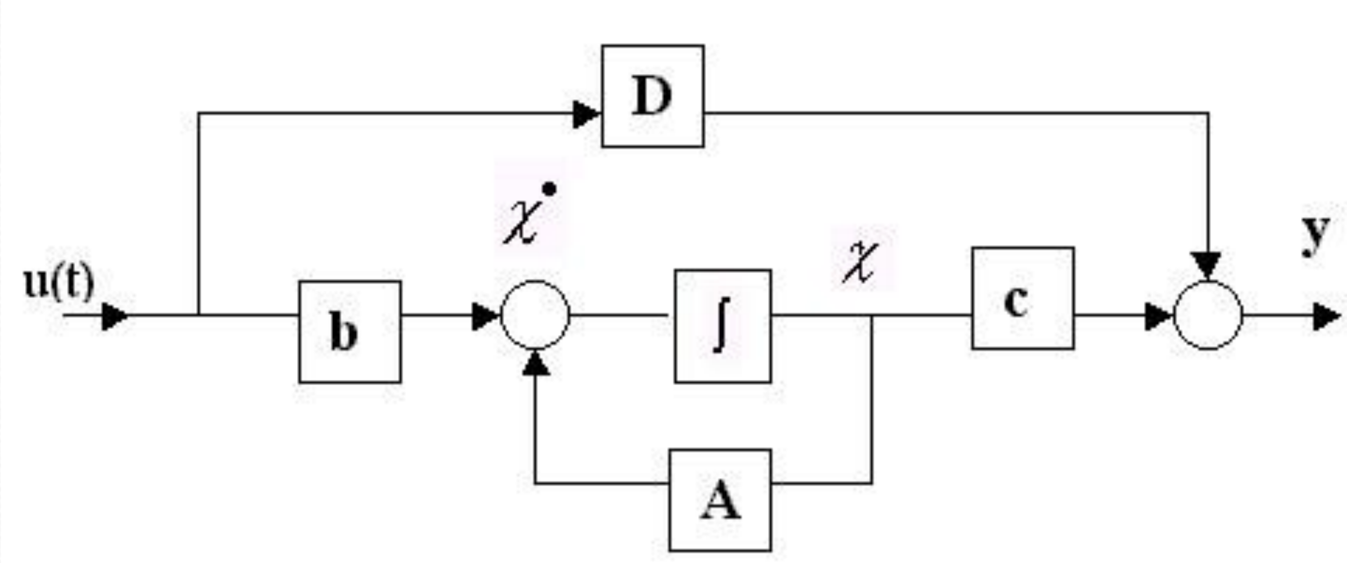
$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}$$

Matrix D has dimensions pxm and it is called the **feedforward** matrix, having the general form

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$$

State Space Equations

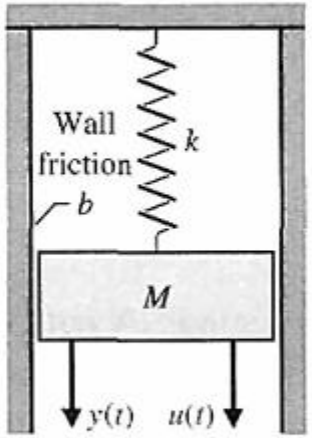
$$SISO \Rightarrow \dot{X} = Ax(t) + Bu(t)$$
$$Y = Cx(t) + Du(t)$$



State Space Representation

- **The general form of a dynamic system**

The concept of a set of state variables that represent a dynamic system can be illustrated in terms of the spring-mass-damper system. A set of state variables sufficient to describe this system includes the position and the velocity of the mass.



- We will define a set of state variables as *(x1, x2),* where

$$x_1(t) = y(t) \quad \text{and} \quad x_2(t) = \frac{dy(t)}{dt}. \qquad \frac{dx_1}{dt} = x_2$$

To write Equation of motion in terms of the state variables, we substitute the state variables as already defined and obtain

$$M\frac{dx_2}{dt} + bx_2 + kx_1 = u(t)$$

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

Therefore, we can write the equations that describe the behavior of the spring-mass damper system as the set of two first-order differential equations

$$\frac{dx_2}{dt} = \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

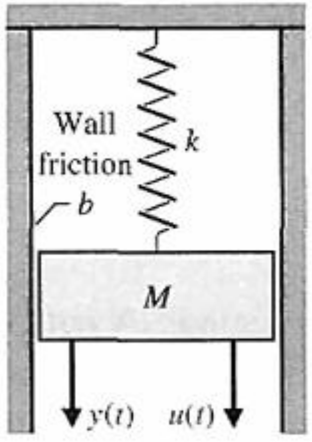
State Space Representation

- **State space matrix**

Let $y = x_1$

$$\frac{dx_1}{dt} = x_2 \qquad \frac{dx_2}{dt} = \frac{-b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{-k}{m} & \frac{-b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\left[u_{(t)}\right]$$



- State space matrix

Let

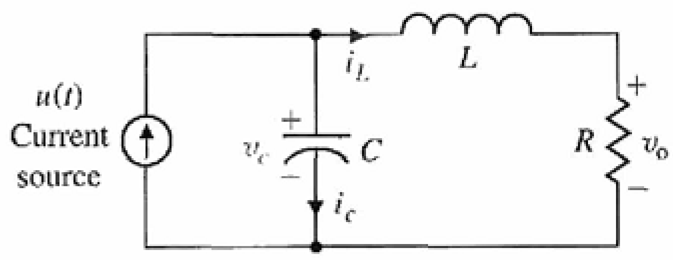
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{-k}{m} & \frac{-b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}\left[u_{(t)}\right]$$

State Space Representation

- **RLC circuit example**
- The state of this system can be described by a set of state variables *(x1, x2),* where *x1* is the capacitor voltage *vc(t)* and *x2* is the inductor current $i_L\{t)$.
- Utilizing Kirchhoff's current law at the junction

$$i_c = C\frac{dv_c}{dt} = +u(t) - i_L$$

OR



Kirchhoff's voltage law for the right-hand loop provides the equation describing the rate of change of inductor current as

$$L\frac{di_L}{dt} = -Ri_L + v_c$$

The output of this system is represented

$$v_o = Ri_L(t)$$

State Space Representation

RLC circuit example

- rewrite Equations as a set of two first-order differential equations in terms of the state variables *x1* and *x2* as follows:

$$\frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t) \qquad \frac{dx_2}{dt} = +\frac{1}{L}x_1 - \frac{R}{L}x_2$$

- The output signal is then

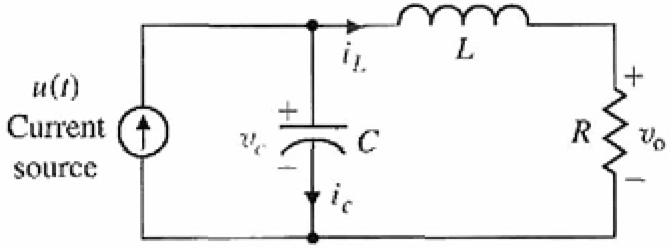
$$y_1(t) = v_o(t) = Rx_2$$

- obtain the state variable differential equation for the *RLC*

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}u(t)$$

- and the output as

$$y = [0 \quad R]\mathbf{x}$$



$$i_c = C\frac{dv_c}{dt} = +u(t) - i_L$$

$$L\frac{di_L}{dt} = -Ri_L + v_c$$

$$v_o = Ri_L(t)$$

TRANSFER FUNCTION FROM THE STATE EQUATION

- Obtain a transfer function *G(s)*, Given the state variable equations. Recalling Equations :where v is the single output and *u* is the single input.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + \mathbf{D}u$$

The Laplace transforms of Equations

$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}U(s)$$

where B is an *n* x 1 matrix, since *u* is a single input, we obtain

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}U(s)$$
$$\mathbf{X}(s) = \Phi(s)\mathbf{B}U(s)$$

- we obtain state transition Matrix

$$[s\mathbf{I} - \mathbf{A}]^{-1} = \Phi(s)$$

TRANSFER FUNCTION FROM THE STATE EQUATION

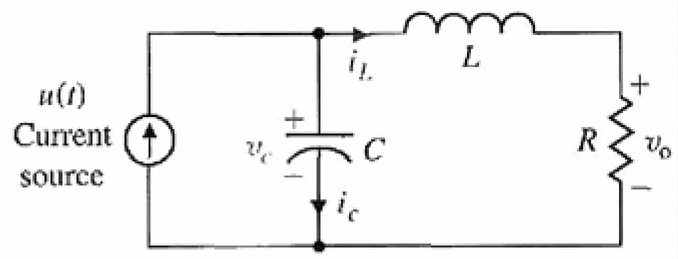
- Transfer function *G(s)*: *G(s)* = *Y(s)*/*U(s)* is

$$G(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}$$

- Let us determine the transfer function *G(s)* = *Y(s)*/*U(s)* for the *RLC* circuit, described by the differential equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}u$$

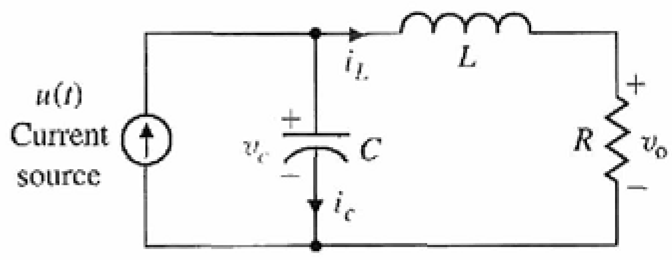
$$y = [0 \quad R]\mathbf{x}.$$



TRANSFER FUNCTION FROM THE STATE EQUATION

- Then we have

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & \frac{1}{C} \\ \frac{-1}{L} & s + \frac{R}{L} \end{bmatrix}$$



- Therefore, we obtain

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{1}{\Delta(s)}\begin{bmatrix} \left(s + \frac{R}{L}\right) & \frac{-1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

$$\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

- Then the transfer function is

$$G(s) = [0 \quad R]\begin{bmatrix} \dfrac{s + \frac{R}{L}}{\Delta(s)} & \dfrac{-1}{C\Delta(s)} \\ \dfrac{1}{L\Delta(s)} & \dfrac{s}{\Delta(s)} \end{bmatrix}\begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} = \frac{R/(LC)}{\Delta(s)} = \frac{R/(LC)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

State Space representation

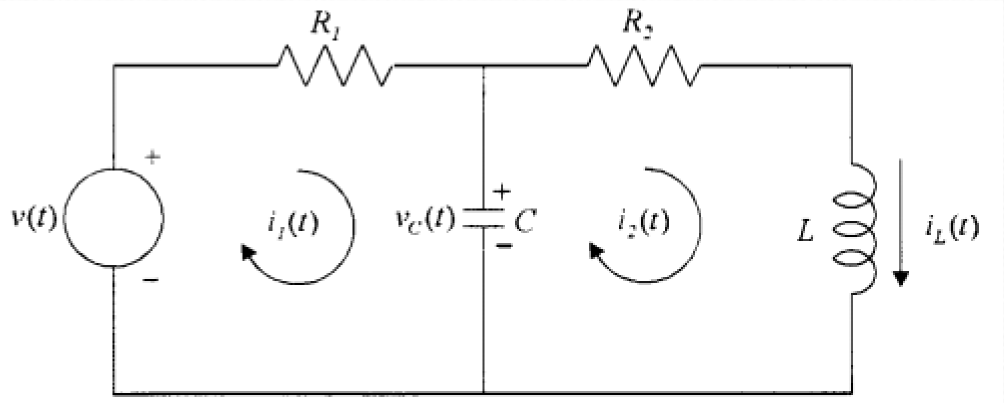
- Transfer from time domain to frequency domain:

$$R_1 i_1(t) + \frac{1}{C}\int_0^t i_1(t)\,\mathrm{d}t - \frac{1}{C}\int_0^t i_2(t)\,\mathrm{d}t = v(t)$$

$$\left[R_1 + \frac{1}{Cs}\right] I_1(s) - \frac{1}{Cs} I_2(s) = V(s)$$

$$-\frac{1}{C}\int_0^t i_1(t)\,\mathrm{d}t + R_2 i_2(t) + L\frac{\mathrm{d}i_2}{\mathrm{d}t} + \frac{1}{C}\int_0^t i_2(t)\,\mathrm{d}t = 0$$

$$-\frac{1}{Cs} I_1(s) + \left[R_2 + Ls + \frac{1}{Cs}\right] \mathrm{I}_2(s) = 0$$



- Transfer function

$$\frac{I_2(s)}{V(s)} = \frac{Cs}{(R_1 Cs + 1)(LCs^2 + R_2 Cs + 1) - 1} = \frac{1}{R_1 LCs^2 + (R_1 R_2 C + L)s + R_1 + R_2}$$

State Space representation

$$\begin{cases} e(t) - R_1\, i_1(t) - L_1 \dfrac{di_1}{dt} - V_C(t) = \phi \\ V_C(t) - L_2 \dfrac{di_2}{dt} - R_2 i_2 = \phi \\ i_c = i_1 - i_2 = C\dfrac{dv_c}{dt} \end{cases}$$

$$x = (i_1 \quad i_2 \quad v_c)^T$$

$$X^{\bullet} = \begin{pmatrix} \dfrac{-R_1}{L_1} & 0 & \dfrac{-1}{L_1} \\ 0 & \dfrac{-R_2}{L_2} & \dfrac{1}{L_2} \\ \dfrac{1}{C} & \dfrac{-1}{C} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \dfrac{1}{L_1} \\ 0 \\ 0 \end{pmatrix} e(t)$$

$$y(t) = (0 \quad R_2 \quad 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

State Space representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Laplace

$$\begin{cases} S\,x(s) - x(0) = Ax(s) + Bu(s) \\ Y(s) = C\,x(s) + Du(s) \end{cases}$$

$$X(S) = (SI - A)^{-1} Bu(s)$$
$$y(s) = (C(SI - A)^{-1} B + D)u(s)$$

Laplace

$$G(S) = C(SI - A)^{-1} B + D$$

$$G(S) = \frac{Q(S)}{|SI - A|}$$

State Space representation

$$\begin{cases}\dot{x}_1 = -5x_1 - x_2 + 2u \\ \dot{x}_2 = 3x_1 - x_2 + 5u\end{cases} \Rightarrow \dot{x} = \begin{pmatrix}-5 & -1 \\ 3 & -1\end{pmatrix} x + \begin{pmatrix}2 \\ 5\end{pmatrix} u$$

$$y = x_1 + 2x_2$$

$$y = (1 \quad 2)\, x$$

$$(SI - A) = \begin{pmatrix}2+5 & 1 \\ -3 & S+1\end{pmatrix}$$

$$(SI - A)^{-1} = \frac{1}{\underbrace{(S+5)(S+1)+3}_{\Delta = (S+2)(S+4)}} \begin{pmatrix}S+1 & -1 \\ 3 & S+5\end{pmatrix}$$

$$G(S) = [1 \quad 2] \frac{1}{\Delta} \begin{bmatrix}S+1 & -1 \\ 3 & S+5\end{bmatrix} \begin{bmatrix}2 \\ 5\end{bmatrix}$$

$$G(s) = \frac{12S + 59}{(S+2)(S+4)}$$

Model Examples

- Pulse Width Modulation (PWM)